Benha University
Faculty of Engineering- Shoubra
Mechanical Engineering Department
Second year Mech. Eng(Power)

Final term exam
Date: 14/1/ 2016
Code: EMP 231
Mathematics and Computers(A)
Duration : 3 hours

Answer all the following questions | No. of questions : **Two** | Total Mark: **70**

Question 1 **[35 marks]**

(a) Show that $v(x,y)=y^3-3x^2y$ is harmonic and find $u(x,y)$ such that $f(z)=u+iv$ is analytic, express $f(z)$ in terms of Z only **[10 marks]**

(b) determine the residues of $f(z)=\frac{4-3z}{z^2-z}$ **[5 marks]**

(c) Prove $f(z)=\sin z\cos z$ is differentiable at any point **[5 marks]**

(d) Evaluate the following integrals: **[15 marks]**

$(i)\ \oint_c (z+\sin z)^{20}\,dz \quad ;\ c\ is\ |z|=5$

$(ii)\ \oint_c \frac{z^2}{(1+\cos z)^3}\,dz \quad ;\ c\ is\ |z-3i|+|z-11i|=10$

$(iii)\ \oint_c \frac{4-3z}{z(z-1)^2}\,dz \quad ;\ c\ is\ |z-1|=\frac{1}{2}$

Question 2 **[35 marks]**

(a) Find the Fourier series of the function **[10 marks]**

$$f(x)=\begin{cases}-x-\pi & ;-\pi\le x\prec 0\\ x & ;0\le x\le\pi\end{cases}$$; And f(x+2π) =f(x) ,then find $\sum_{n=1}^{n=\infty}\frac{1}{(2n-1)^2}$

(b) Expand the function f(x) = x ; x in [0, π/2] and f(x+2 π) = f(x) **[10 marks]**
in Even Cosine Harmonic

(c) Solve the following partial differential equation: **[15 marks]**

$$u_{tt}-4u_{xx}=0 \quad ;\ 0\prec x\prec 1,\ t\succ 0$$
$$B.C\ :u(0,t)=u(1,t)=0$$
$$I.C\ :u(x,0)=x+1\ ;\ u_t(x,0)=x$$

Good Luck *Dr.Eng. Zaki Ahmed Zaki*

Second year Mech. Engineering (power) final term Exam (Jan. 2016)

Model Answer

[1] **(a) Show that** $v(x,y) = y^3 - 3x^2y$ **is harmonic and find**

$u(x,y)$ **such that** $f(z) = u + iv$ **is analytic, express** $f(z)$ **in terms of** Z **only** **[10 marks]**

$v(x,y) = y^3 - 3x^2y \rightarrow v_x = -6xy \;, v_{xx} = -6y \;, v_y = 3y^2 - 3x^2 , v_{yy} = 6y$

$\because v_{xx} + v_{yy} = 6y - 6y = 0$; *then v is harmonic*

$\because f(z)$ *is analytic then* $u_x = v_y \rightarrow u = \int v_y \, dx = \int 3y^2 - 3x^2 \, dx$

$u = 3y^2x - x^3 + c(y); \; \because u_y = -v_x \rightarrow 6yx + c'(y) = 6xy$

$c'(y) = 0 \rightarrow c(y) = constant$ *, may be neglected*

$\therefore u(x,y) = 3y^2x - x^3 \rightarrow f(z) = u + iv = 3y^2x - x^3 + i(y^3 - 3x^2y)$; *let* $y = 0, x = z$

$f(z) = -z^3$

(b) determine the residues of $f(z) = \frac{4-3z}{z^2-z}$ **[5 marks]**

$$Res_{z=0} = \frac{1}{0!} \lim_{z \to 0} [\frac{z(4-3z)}{z(z-1)}] = -4$$

$$Res_{z=1} = \frac{1}{0!} \lim_{z \to 1} [\frac{(z-1)(4-3z)}{z(z-1)}] = 1$$

(c) Prove $f(z) = \sin z \cos z$ **is analytic.** **[5 marks]**

f(z)= 0.5sin 2z = 0.5 sin(2x+i2y) =0.5(sin 2x cosi2y + cos2x sini2y)
= 0.5(sin 2x cosh 2y + i cos2 x sinh2 y)

Real part (u) =0.5(sin 2x cosh2 y); imaginary part (v) =0.5 cos2 x sinh 2y

$u_x = \cos 2x \cosh 2y$; $u_y = \sin 2x \sinh 2y$

$v_x = -\sin 2x \sinh 2y$; $v_y = \cos 2x \cosh 2y$, **then** $u_x = v_y$ *and* $u_y = -v_x$

So, $f(z) = \sin z \cos z$ **is analytic(differentiable at any point)**

(d) Evaluate the following integrals: [<u>15 marks</u>]

$$(i)\oint_c (z+\sin z)^{20}\,dz \quad ; c\ is\ |z|=5$$

The function is analytic (defined) inside the region then

$$\oint_c (z+\sin z)^{20}\,dz = 0$$

$$(ii)\oint_c \frac{z^2}{(1+\cos z)^3}\,dz \quad ; c\ is\ |z-3i|+|z-11i|=10$$

The function is not defined at $z=\pi$ which is outside this region $[|\pi-3i|+|\pi-11i|]\succ 10$

then $(ii)\oint_c \frac{z^2}{(1+\cos z)^3}\,dz = 0$

(iii) $\oint_c \frac{4-3z}{z(z-1)^2}\,dz \quad ; c\ is\ |z-1| = \frac{1}{2}$

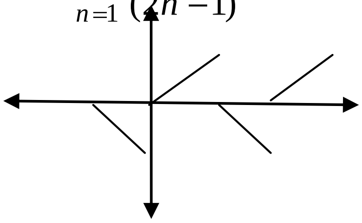
The function is not defined at z = 0(outside) and z=1 (inside) the region then

$$\oint_c \frac{4-3z}{z(z-1)}dz = \oint_c \frac{(\frac{4-3z}{z})}{(z-1)^2}dz = 2\pi i\,\frac{(-3z-4+3z)}{z^2} = 2\pi i\,(\frac{-4}{1}) = -8\pi i$$

Question 2

(a) Find the Fourier series of the function [<u>10 marks</u>]

$$f(x) = \begin{cases} -x-\pi & ;-\pi \le x \prec 0 \\ x & ;0 \le x \le \pi \end{cases}$$; And f(x+2π) =f(x) then find $\sum_{n=1}^{n=\infty}\frac{1}{(2n-1)^2}$



The given function is **odd harmonic** then

$$a_{2n-1} = \frac{2}{\pi}\int_0^{\pi} x\cos(2n-1)x\,dx = \frac{-4}{\pi(2n-1)^2} \quad ; b_{2n-1} = \frac{2}{\pi}\int_0^{\pi} x\sin(2n-1)x\,dx = \frac{2}{(2n-1)}$$

$$then\ f(x) = \sum_{n=1}^{n=\infty}\left[\frac{-4}{\pi(2n-1)^2}\cos(2n-1)x + \frac{2}{2n-1}\sin(2n-1)x\right] \quad ;$$

$$at\ x=0 \quad ; \frac{0+-\pi}{2} = \sum_{n=1}^{n=\infty}\left[\frac{-4}{\pi(2n-1)^2}\cos(2n-1)0 + \frac{2}{2n-1}\sin(2n-1)0\right]$$

$$\frac{\pi^2}{8} = \sum_{n=1}^{n=\infty}\frac{1}{(2n-1)^2}$$

(b) Expand the function f(x) = x ; x in [0, π/2] and f(x+2 π) = f(x) **[10 marks]**

in Even cosine harmonic

$$a_0 = \frac{4}{\pi}\int_0^{\pi/2} x\,dx = \pi/2$$

$$a_{2n} = \frac{4}{\pi}\int_0^{\pi/2} x\cos 2nx\,dx \;, a_{2n} = \frac{4}{\pi}\left[\frac{x\,sin\,2nx}{2n} + \frac{cos2nx}{4n^2}\right]$$

$$= \frac{\cos(n\pi) - 1}{\pi n^2}$$

$$f(x) = \pi/4 + \sum_1^{\infty}\left(\frac{cosn\pi - 1}{\pi n^2}\right)cos2nx$$

(c) [10 marks]

$$u_{tt} - 4u_{xx} = 0 \quad ; 0 \prec x \prec 1, t \geq 0$$

$$B.C \;: u(0,t) = u(1,t) = 0$$

$$I.C \;: u(x,0) = x + 1 \;;\; u_t(x,0) = x$$

$$u_{tt} - c^2 u_{xx} = 0 \quad ,then\;\; c = 2\,; L = 1\,; f(x) = x + 1\,; g(x) = x + 1;$$

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\;dx = \frac{2}{1}\int_0^1 (x+1)\sin n\pi x\,dx = \frac{2}{n\pi}[2(-1)^{n+1} + 1]$$

$$B_n = \frac{2}{cn\pi}\int_0^L g(x)\sin\frac{n\pi}{L}x\;dx = \frac{2}{2n\pi}\int_0^1 x\sin n\pi x\;dx = \frac{(-1)^{n+1}}{(n\pi)^2}\;\; then;$$

$$u(x,t) = \sum_{n=1}^{n=\infty}\sin\frac{n\pi x}{L}\left(A_n\cos\frac{cn\pi t}{L} + B_n\sin\frac{cn\pi t}{L}\right)$$

Good Luck *Dr.Eng. Zaki A. Zaki*