Question 3 **[25]**

a) For what values of x is the function $f(x) = \left(\dfrac{e^{\sin x}}{4 - \sqrt{x^2 - 9}}\right)$ continuous?. **[6]**

b) Use the limit definition to compute the first derivative for $f(x) = 5x^2 - 3x + 7$. **[7]**

c) Find the first derivative for the following functions **[12]**

i) $y(x) = (3x^2 + 1)^{1/x} + \dfrac{[\ell n\, x]^x}{2^{3x^5 - 1}}$, $g(x) = \dfrac{x \csc x}{3 - \csc x} + x^2 \sec^2(\pi x)$

Question 4 **[25]**

a) Evaluate the following limits

i) $\lim\limits_{x \to 0} \dfrac{3^x - 2^x}{x^2 - x}$, ii) $\lim\limits_{x \to 0^+} (\tan x)^{x^2}$ **[6]**

b) Expand $f(x) = f(x) = x^2 \cos x$ using Taylor Maclaurin series **[7]**

c) If the sum of 2 numbers is k, find the minimum sum of their squares **[6]**

d) Evaluate i) $\int \left(\dfrac{1 + \ell n\, x}{5 + x\, \ell n\, x}\right) dx$, ii) $\int \dfrac{1}{x^3}\left(7 + \dfrac{5}{x}\right)^{-3} dx$ **[6]**

Model answer

Q3- a) For the function f(x) to be defined $4-\sqrt{x^2-9}\neq 0$, therefore f(x) is continuous at $x\leq -3$, or $x\geq 3$ except at x = -5 or x = -5

b) $f`(x) = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$

$$= \lim_{\Delta x\to 0}\frac{[5(x+\Delta x)^2 - 3(x+\Delta x) + 7] - [5x^2 - 3x + 7]}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{5[(2x\Delta x+(\Delta x)^2] - 3(\Delta x)}{\Delta x} = 10x - 3$$

c-i) $y(x) = (3x^2+1)^{1/x} + \frac{[\ell n\, x]^x}{2^{3x^5-1}} \Rightarrow y`(x) = (3x^2+1)^{1/x}\left[\frac{-\ell n(3x^2+1)}{x^2} + \frac{6}{(3x^2+1)}\right]$

$$+ \frac{[\ell n\, x]^x[\ell n(\ell n x) + x(\frac{1/x}{\ln x})][2^{3x^5-1}] - [\ell n\, x]^x[15x^4][2^{3x^5-1}]\,\ell n\, 2}{[2^{3x^5-1}]^2}$$

ii) $g(x) = x^2\sec^2(\pi x) + \frac{x\csc x}{3-\csc x} \Rightarrow g`(x) = 2x^2\pi[\sec^2(\pi x)\tan(\pi x)] + 2x\sec^2(\pi x) +$

$$\frac{[-x\csc x\cot x + \csc x][3-\csc x] - [x\csc x][\csc x\cot x]}{(3-\csc x)^2}$$

Q4-a-i) $\lim_{x\to 0}\frac{3^x-2^x}{x^2-x} = \frac{0}{0} = \lim_{x\to 0}\frac{3^x\,\ell n\,3 - 2^x\,\ell n\,2}{2x-1} = \ell n\,2 - \ell n\,3$

ii) This limit of the indeterminate form 0^0. Let $y = (\tan x)^{x^2} \Rightarrow \ell n\, y = x^2\,\ell n(\tan x)$ and we have to evaluate $\lim_{x\to 0^+}\ell n\, y = \lim_{x\to 0^+} x^2\,\ell n(\tan x)$.

This intermediate form is $0\bullet\infty$ and so rewrite the above limit such that $\lim_{x\to 0^+} x^2\ell n(\tan x) = \lim_{x\to 0^+}\frac{\ell n(\tan x)}{1/x^2} = \frac{-\infty}{\infty}$, then L'Hospital's rule can be used to evaluate this limit $\lim_{x\to 0^+}\frac{\ell n(\tan x)}{1/x^2} = -\lim_{x\to 0^+}\frac{x^3\sec^2 x}{2\tan x} = \frac{0}{0}$

$$= -\lim_{x\to 0^+}\frac{3x^2\sec^2 x + 2x^3\sec^2 x\tan x}{2\sec^2 x} = 0.$$

Therefore $\lim_{x\to 0^+} (\tan x)^{x^2} = \lim_{x\to 0^+} e^{\ell n\, y} = e^{\lim_{x\to 0^+} \ell n\, y} = e^0 = 1$

b) Let g(x) = cos x $\Rightarrow$ g\`(x) = - sin x, g\`\`(x) = - cos x, g\`\`\`(x) = sin x,….,therefore

g(0) = 1, g\`(0) = 0, g\`\`(0) = -1, g\`\`\`(0) = 0,…..

Substitute in above equation, thus cos x = $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$

After we expand cos x, then multiply by x^2, we get $x^2 \cos x = x^2 - \frac{x^4}{2!} + \frac{x^6}{4!} - \dots$

c) Let the sum of the two numbers is expressed by x + y = k, therefore the sum of the two squares is expressed by $S = x^2 + y^2$. To get the minimum sum of their squares, $\frac{dS}{dx} = 0 \Rightarrow 2x - 2(k - x) = 0 \Rightarrow x = \frac{k}{2}$ and $y = \frac{k}{2} \Rightarrow \frac{d^2S}{dx^2} = 4,$ hence the minimum sum of their squares $S = x^2 + y^2 = \frac{k^2}{2}$.

d) i) $\int (\frac{1 + \ell n\, x}{5 + x\, \ell n\, x}) dx = \ell n\, (5 + x\, \ell n\, x) + c$

ii) $\int \frac{1}{x^3} (7 + \frac{5}{x})^{-3} dx = \int [x\, [7 + \frac{5}{x}]\ ]^{-3}\, dx = \frac{1}{7} \int [7x + 5]^{-3} (7) dx = -\frac{1}{14[7x + 5]^2} + c$