Benha University Faculty of Engineering- Shoubra (Civil تخلفات)Engineering Department		Final term exam Date: 14th of June 2014 Mathematics & Statistics Code: EMP 151 Duration : 3 hours
Answer all the following questions	No. of questions : 4	Total Mark: 70 marks

Q1 Solve the differential equations [15 marks]

$(x^3 + y^3)dx - 3xy^2\, dy = 0$ $\quad$ $y'' - 3y' + 2y = 2x^2 + e^x + xe^x$ $\quad$ $(D^2 - 6D + 13)y = 8e^{3x}\cos 2x$

Q2 Test for convergence [15 marks]

$$\sum_{n=1}^{\infty} \frac{5^n + 7^n}{3^n + 2^n} \qquad \sum_{n=1}^{\infty} \frac{(5n)^n}{(9n+11)^n} \qquad \sum_{n=1}^{\infty} \frac{\cos(n\pi)}{\sqrt{n}}$$

Q3 Answer the following [25 marks]

I) At Kennedy middle School, the probability that a student takes Technology and Spanish is **0.087** and the probability that a student takes Technology is **0.68**. What is the probability that a student takes Spanish given that the student is taking Technology?

II) If a r.v. X takes the values 1, 2, 3, 4 such that $2P(X = 1) = 5P(X = 2) = P(X = 3) = 3P(X = 4)$. Find the probability distribution of X, also find mean and variance**.**

III) Let X be a continuous r.v. with p.d.f. $f(x) = (1/x)$ for $1 < x < e$, find $E(\ln X)$, $Var(X)$, median, $P(1.5 > x)$, $P(1.2 < x)$.

Q4 Answer the following [15 marks]

I) Find the point on the plane $3x + 2y + z = 24$ that is nearest to the origin

II) Find envelope for the family of ellipses $\dfrac{x^2}{c^2} + \dfrac{y^2}{(1-c)^2} - 1 = 0$

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Model answer

Answer of question 1

$(x^3 + y^3)dx - 3xy^2\,dy = 0$ (solve)

Answer

It is homogeneous, thus put $y = vx \Rightarrow dy = v\,dx + x\,dv$

Thus $(1-2v^2)\,dx - 3v^2 x\,dv = 0 \Rightarrow \frac{dx}{x} - \frac{3v^2}{1-2v^3}dv = 0 \Rightarrow \ln x + \frac{1}{2}\ln(1-2v^3) = c$

$y'' - 3y' + 2y = 2x^2 + e^x + xe^x$

Answer

$y(x) = y_p + y_H$, therefore $y_H = c_1 e^{2x} + c_2 e^x$

$$y_p = \frac{1}{D^2-3D+2}2x^2 + \frac{1}{D^2-3D+2}e^x + \frac{1}{D^2-3D+2}xe^x$$

$$y_p = \frac{1}{2[1+\frac{D^2-3D}{2}]}2x^2 + \frac{1}{(D-1)(D-2)}e^x + e^x\frac{1}{(D+1)^2-3(D+1)+2}x$$

$$y_p = [1-(\frac{D^2-3D}{2})+(\frac{D^2-3D}{2})^2+\ldots]x^2 - \frac{1}{(D-1)}e^x + e^x\frac{1}{D^2-D}x$$

$$y_p = [1+\frac{7D^2}{4}+\frac{3D}{2}+\ldots]x^2 - xe^x - \frac{e^x}{2}[1+D+D^2]x$$

$$y_p = 7/2 + 4x - 2xe^x - e^x - \frac{e^x}{2}x^2$$

$(D^2 - 6D + 13)y = 8e^{3x}\cos 2x$

Answer

$y_h = e^{3x}[c_1\cos 2x + c_2\sin 2x]$,

$y_p = \frac{1}{D^2 - 6D + 13} 8e^{3x}\cos 2x = 8e^{3x}\frac{1}{(D+3)^2 - 6(D+3) + 13}\cos 2x =$

$8e^{3x}\frac{1}{D^2+4}\cos 2x = 2xe^{3x}\sin 2x$

Answer of question 2

$\sum_{n=1}^{\infty}\frac{5^n + 7^n}{3^n + 2^n}$ **(Test for convergence)**

Answer

By ratio test, we get that $\lim_{n\to\infty}\left(\frac{5^{n+1}+7^{n+1}}{3^{n+1}+2^{n+1}}\right)\left(\frac{3^n+2^n}{5^n+7^n}\right) = \lim_{n\to\infty}\frac{7^{n+1}}{3^{n+1}}\left[\frac{(5/7)^{n+1}+1}{1+(2/3)^{n+1}}\right]$ $\frac{3^n}{7^n}\left[\frac{3^n+2^n}{5^n+7^n}\right]$ = 7/3 > 1, therefore the series is divergent.

$\sum_{n=1}^{\infty}\frac{(5n)^n}{(9n+11)^n}$ **(Test for convergence)**

Answer

Since $\lim_{n\to\infty}\sqrt[n]{\left(\frac{5n}{9n+11}\right)^n} = \lim_{n\to\infty}\left(\frac{5n}{9n+11}\right) = \frac{5}{9} < 1$, therefore $\sum_{n=1}^{\infty}\frac{(5n)^n}{(9n+11)^n}$ is convergent

$\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{\sqrt{n}}$ **(Test for convergence)**

Answer

Let $U_n = \frac{1}{\sqrt{n}}$, $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$, $U_{n+1} = \frac{1}{\sqrt{n+1}}$, hence $U_n > U_{n+1}$. By using integral test, we will get that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent, so $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{\sqrt{n}}$ is called conditionally convergent.

Answer of question 3

I) $P(T\cap S) = 0.087$, $P(T) = 0,68$, therefore $P(S/T) = P(T\cap S) / P(T) = 0.087/0.68$

II) Let $2P(X = 1)= 3P(X = 2) = P(X = 3) = 5P(X = 4) = 30\,P$, therefore $P(X=1) = 15P$, $P(X=2) = 10P$, $P(X = 3) = 30P$, $P(X = 4) = 6P$, but $P(X=1) + P(X=2) + P(X=3) + P(X=4) = 1$, thus $P = 1/61$

X	1	2	3	4
fX)	15/61	10/61	30/61	6/61

$E(X) = (1/61)[15 + 20 + 90 + 24] = 149/61$

$E(X^2) = (1/61)[15 + 40 + 270 + 96] = 421/61$

$Var(X) = (421/61) – (149/61)^2$

III) $E(\text{Ln}x) = \int_1^e \text{Ln}\, x\, (\frac{1}{x})\, dx = \frac{(\ln x)^2}{2} = \frac{1}{2}$ $E(x) = \int_1^e x(\frac{1}{x})dx = \int_1^e dx = e - 1$

$E(x^2) = \int_1^e x^2(\frac{1}{x})dx = \int_1^e x\, dx = \frac{x^2}{2} = \frac{e^2-1}{2}$

$Var(X) = E(X^2) - E(X)^2 = \frac{e^2-1}{2} - \frac{2(e^2-2e+1)}{2} = \frac{4e - e^2 - 3}{2}$

$\int_1^x \frac{1}{x} dx = 0.5 \Rightarrow \ln x = 0.5 \Rightarrow x = 1.64872$ is the median

$P(1.5 > x) = \int_{1}^{1.5} \frac{1}{x} dx = \ln(1.5) = 0.4055$ $P(1.2 < x)$

$$= \int_{1.2}^{e} \frac{1}{x} dx = 1 - \ln(1.2) = 0.8177$$

Answer of question 4

I) $f(x, y, z) = x^2 + y^2 + z^2$ s.t $g(x, y, z) = 3x + 2y + z = 24$, therefore $f_x = \lambda g_x \Rightarrow 2x = 3\lambda$ and $f_y = \lambda g_y \Rightarrow 2y = 2\lambda$ and $f_z = \lambda g_z \Rightarrow 2z = \lambda$, thus $(2/3)x = y = 2z$, hence $y = (2/3)x$ and $z = (1/3)x$, but $3x + 2y + z = 24$ and so $3x + 2(2/3)x + (1/3)x = 24 \Rightarrow 14x = 72 \Rightarrow x = 36/7$ and $y = 24/7$ and $z = 12/7$, therefore $(36/7, 24/7, 12/7)$ is the nearest point

II) $\frac{\partial}{\partial c}\left[\frac{x^2}{c^2} + \frac{y^2}{(1-c)^2} - 1 = 0\right] \Rightarrow \frac{-2x^2}{c^3} + \frac{2y^2}{(1-c)^3} = 0 \Rightarrow \frac{(1-c)^3}{c^3} = \frac{y^2}{x^2} \Rightarrow c = \frac{1}{\sqrt[3]{\frac{y^2}{x^2}} + 1}$

The envelope is $\frac{x^2}{\left[\frac{1}{\sqrt[3]{\frac{y^2}{x^2}} + 1}\right]^2} + \frac{y^2}{\left(1 - \left[\frac{1}{\sqrt[3]{\frac{y^2}{x^2}} + 1}\right]\right)^2} - 1 = 0$