Benha University Faculty of Engineering – Shoubra Department of Energy and Sustainable Energy Course: Mathematics 4 Code: EMP 202		Final Exam Date: June 10, 2015 Duration: 3 hours Answer **All** questions
The exam consists of one page	No. of questions: 4	Total Mark: 40

<u>Question 1</u>

Solve the following equations: 9

(a) $y \sin x \; dx + (1 + y^2)dy = 0$ (b) $(y + x \cos y)dy + (x + \sin y)dx = 0$

(c) $y`` - 2y` - 3y = e^{-x} + 3^x$ (d) $y`` + y` = 3 + x^2$

(e) $(D^2 + 4)y = 1 + \cos 2x$ (f) $(D^2 + 1)y = \tan x$

<u>Question 2</u>

(a)Prove that: If f(t) is function with Laplace transformation F(s). Then 2

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds$$

(b)Find Laplace transformation of the following: 4

(i) $f(t) = 3 + t^3 + \sinh 3t$ (ii) $f(t) = \sin 3t + t \cos t$

(iii) $f(t) = \sin(t - 3)\,,\; t > 3$ (iv) $f(t) = \frac{e^{3t} - e^{2t}}{t}$

(c)Find the inverse Laplace transform of: (i) $F(s) = \frac{s}{S^2 - 2s - 3}$ (ii)$F(s) = \tan^{-1}(s - 2)$ 2

(d)Solve the equation: $y`` + 2y` + y = e^{-t}$, $y(0) = 0$ , $y`(0) = 1$. 3

<u>Question 3</u>

(a)Find the root of the equation $x^3 + 3x - 10 = 0$ by usingNewton method. 10

(b)Construct the difference table to the following data.

Hence find interpolation polynomial interpolate the function y = f(x)

at the points (0, 1), (0.1, 1.32), (0.2, 1.68), (0.3, 2.08), (0.4, 2.52).

<u>Question 4</u>

(a)Approximate the integrals $\int_0^1 \sqrt{1+x}\,dx$ using Sampson's rule. Estimate the error 10

by computing the exact value.

(b)Solve the differential equation : $y' = x + y\,,\;\; 0 < x < 1,\;\; y(0) = 1$

using Euler method considering $h = 0.2$. Estimate the error at $x = 0.4$ by comparing your result.

Good Luck *Dr. Mohamed Eid* *Dr. Fathi Abdsallam*

Answer

Question 1

(a) $y \sin x \ dx + (1 + y^2)dy = 0$

Divide by y, we get : $\sin x \ dx + (\frac{1}{y} + y)dy = 0$

Then, the solution is : $-\cos x + \ln y + \frac{1}{2}y^2 = c$

(b) $(y + x\cos y)\mathrm{dy} + (\mathrm{x} + \sin \mathrm{y})\mathrm{dx} = 0$

Since $P = x + \sin y$, $q = y + x\cos y$ and $P_y = \cos y = q_x$. The equation is exact.

Then $\int(\mathrm{x} + \sin \mathrm{y})\mathrm{dx} = \frac{1}{2}x^2 + x\sin y$, $\int(y + x\cos \mathrm{y})\mathrm{dy} = \frac{1}{2}y^2 + x\sin y$

Then, the solution is : $\frac{1}{2}x^2 + \frac{1}{2}y^2 + x\sin y = c$

(c) $y`` - 2y` - 3y = e^{-x} + 3^x$

The A.E is : $k^2 - 2k - 3 = (k-3)(k+1) = 0$. Then k = 3, – 1.

Then $y_c = Ae^{3x} + Be^{-x}$

$$y_p = \frac{1}{(D-3)(D+1)}e^{-x} + \frac{1}{D^2-2D-3}e^{(\ln 3)x} = \frac{1}{-4}x.e^{-x} + \frac{1}{(\ln 3)^2 - 2\ln 3 - 3}e^{(\ln 3)x}$$

$y = y_c + y_p$

(d) $\mathrm{y}`` + \mathrm{y}` = 3 + \mathrm{x}^2$

The A.E is : $k^2 + k = k(k+1) = 0$. Then k = 0, – 1.

Then $y_c = A + Be^{-x}$

$$y_p = \frac{1}{D(D+1)}(3 + x^2) = \frac{1}{D}(1 + D)^{-1}(3 + x^2) = \frac{1}{D}(1 - D + D^2 + \cdots)(3 + x^2)$$

$$= \frac{1}{D}(3 + x^2 - 2x + 2) = 5x + \frac{1}{3}x^3 - x^2$$

$y = y_c + y_p$

(e) $(\mathrm{D}^2 + 4)\mathrm{y} = 1 + \cos 2\mathrm{x}$

The A.E is : $k^2 + 4 = 0$. Then $k = \pm 2i$.

Then $y_c = A\cos 2x + B\sin 2x$

$$y_p = \frac{1}{D^2+4}(1 + \cos 2x) = \frac{1}{D^2+4}1 + \frac{1}{D^2+4}\cos 2x = \frac{1}{4} + \frac{x}{4}\sin 2x$$

$y = y_c + y_p$

(f) $(D^2+1)y = \tan x$

The A.E is : $k^2+1=0$. Then $k=\pm i$.

Then $y_c = A\cos x + B\sin x$

$$w = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$$w_1 = \begin{vmatrix} 0 & \sin x \\ 1 & \cos x \end{vmatrix} = -\sin x$$

$$w_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & 1 \end{vmatrix} = \cos x$$

Then $y_p = \cos x \int \frac{w_1}{w} \tan x\, dx + \sin x \int \frac{w_2}{w} \tan x\, dx$

$$= \cos x \int \frac{-\sin x}{1} \tan x\, dx + \sin x \int \frac{\cos x}{1} \tan x\, dx$$

$$= \cos x \int (\cos x - \sec x)\, dx + \sin x \int \sin x\, dx$$

$$= \cos x\, (\sin x - \ln(\sec x + \tan x)) + \sin x\, (-\cos x)$$

$y = y_c + y_p$

…………………………………………………………………………………*(9 Marks)*

Question 2

(a)Theorem

…………………………………………………………………………………*(2 Marks)*

(b)(i) $f(t) = 3 + t^3 + \sinh 3t$

$$F(s) = \frac{3}{s} + \frac{3!}{s^4} + \frac{3}{s^2-9}$$

(ii) $f(t) = \sin 3t + t\cos t$

$$F(s) = \frac{3}{s^2+9} - \left(\frac{s}{s^2+1}\right)^{`} = \frac{3}{s^2+9} - \frac{1-s^2}{(s^2+1)^2}$$

(iii) $f(t) = \sin(t-3)\,,\ t > 3$

Since $L\{\sin t\} = \frac{1}{s^2+1}$. Then $F(s) = \frac{1}{s^2+1} e^{-3s}$

(iv) $f(t) = \frac{e^{3t} - e^{2t}}{t}$

Since $L\{e^{3t} - e^{2t}\} = \frac{1}{s-3} - \frac{1}{s-2}$. Then $F(s) = \int_s^\infty \frac{1}{s-3} - \frac{1}{s-2} ds = \ln \frac{s-2}{s-3}$

…………………………………………………………………………………*(4 Marks)* (c)(i) $F(s) = \frac{s}{S^2-2s-3}$

By method of partial fractions, we get $F(s) = \frac{3/4}{s-3} + \frac{1/4}{s+1}$

Then $f(t) = \frac{3}{4}e^{3t} + \frac{1}{4}e^{-t}$

(ii)$F(s) = \tan^{-1}(s-2)$

Since $F`(s) = \frac{1}{(s-2)^2+1}$ and $L^{-1}\left\{\frac{1}{(s-2)^2+1}\right\} = e^{2t}\sin t = -t\,f(t)$

Then $f(t) = -\frac{e^{3t}\sin t}{t}$

………………………………………………………………………………………*(2 Marks)*

(d) $y`` + 2y` + y = e^{-t}$

$L\{y`` + 2y` + y\} = \{e^{-t}\}$

Then $s^2Y - sy(0) - y`(0) + 2(sY - y(0)) + Y = \frac{1}{s+1}$

From the initial conditions, we get

$s^2Y - 1 + 2sY + Y = \frac{1}{s+1}$ or $Y = \frac{s+2}{(s+1)^3} = \frac{1}{(s+1)^2} + \frac{1}{(s+1)^3}$

Then, the solution is : Then $y(t) = t.e^{-t} + \frac{t^2}{2}e^{-t}$

………………………………………………………………………………………*(3 Marks)*

Dr. Mohamed Eid

Question 3 *10 Marks*

(a)Find the root of the equation $x^3 + 3x - 10 = 0$ by using Newton method.

Answer

Newton's method

$$\boldsymbol{x^3 + 3x - 10 = 0}$$

$$\boldsymbol{f(x) = x^3 + 3x - 10}$$

$$\boldsymbol{f'(x) = 3x^2 + 3}$$

$$\boldsymbol{x_{n+1} = x_n - \frac{x_n{}^3 + 3x_n - 10}{3x_n{}^2 + 3} = \frac{2x_n{}^3 + 10}{3x_n{}^2 + 3}}$$

i	x_i	x_{i+1}	$f(x_{i+1})$
0	1	2	4

1	2	1.733333333	0.407703704
2	1.733333333	1.699395733	0.005950068
3	1.699395733	1.698885604	1.32658E-06
4	1.698885604	1.69888549	6.39488E-14
5	1.69888549	1.69888549	0

$\boldsymbol{R = 1.69888549}$

(b)Construct the difference table to the following data.
Hence find interpolation polynomial interpolate the function $y = f(x)$ atthese points (0, 1), (0.1, 1.32), (0.2, 1.68), (0.3, 2.08), (0.4, 2.52).

Answer

The difference table

x	y	∂f	$\partial^2 f$	$\partial^3 f$	$\partial^4 f$
0	**1**	**0.32**	**0.02**	**0.04**	**-0.06**
0.1	**1.32**	**0.34**	**0.06**	-0.02	
0.2	**1.68**	**0.4**	0.04		
0.3	**2.08**	0.44			
0.4	**2.52**				

$$P_3(x) = y_0 + \partial f\,(x - x_0) + \partial^2 f\,(x - x_0)(x - x_1)$$
$$+ \partial^3 f\,(x - x_0)(x - x_1)(x - x_2) + \partial f\,(x - x_0)(x - x_1)(x - x_2)(x - x_3)$$
$$= 1 + 0.32(x - 0) + 0.02(x - 0)(x - 0.1) + 0.04(x - 0)(x - 0.1)(x - 0.2)$$
$$- 0.06(x - 0)(x - 0.1)(x - 0.2)(x - 0.3)$$

Question (4) *10 Marks*

(b)Approximate the integrals $\int_0^1 \sqrt{1+x}\,dx$ using Sampson's rule. Estimate the error by computing the exact value.

Answer

$$\int_0^1 \sqrt{1+x}\,dx$$

Solution: (a) by Trapezoidal Rule

i	x_i	$f(x_i)$	$2\,f(x_i)$
0	0	1	1
1	0.1	1.048808848	2.097617696
2	0.2	1.095445115	2.19089023
3	0.3	1.140175425	2.28035085
4	0.4	1.183215957	2.366431913
5	0.5	1.224744871	2.449489743
6	0.6	1.264911064	2.529822128
7	0.7	1.303840481	2.607680962
8	0.8	1.341640786	2.683281573
9	0.9	1.378404875	2.75680975
10	1	1.414213562	1.414213562

I= 1.21882942

The exact value $\int_0^1 \sqrt{1+x}\,dx = \frac{2}{3}\left[(1+x)^{3/2}\right]_0^1 = \frac{2}{3}\left[2^{3/2}-1\right] = 1.21895141649764$

Error=1.22E-4

(b)Solve the differential equation $y' = x + y,\quad 0 < x < 1,\quad y(0) = 1$ using Euler method considering $h = 0.2$. Estimate the error at $x = 0.4$ by comparing your result with the exact solution of the problem.

Answer

$$y' = (x + y),\qquad 0 < x < 1,\qquad y(0) = 1$$

Answer:

$$y_{n+1} = y_n + hf(x_n, y_n)$$

$$\begin{aligned} y_{n+1} &= y_n + hf(x_n, y_n) \\ &= y_n + 0.2[x_n + y_n)] \\ &= y_n + 0.2[0.2n + y_n)] \end{aligned}$$

$$y_{n+1} = y_n + (0.2)^2 n + 0.2y_n)$$

$$y_{n+1} = 1.2y_n + (0.2)^2 n$$

$$\boxed{n = 0,1,2,3,4,\ ...}$$

$$y_1 = 1.2y_0 + (0.04)(0) = 1.2(1) + 0 = 1.2$$

$$y_2 = 1.2y_1 + (0.04)(1) = 1.2(1.2) + (0.04)(1) = 1.48$$

$$y_3 = 1.2y_2 + (0.04)(2) = 1.2(1.48) + (0.04)(2) = 1.856$$

$$y_4 = 1.2y_3 + (0.04)(3) = 1.2(1.856) + (0.04)(3) = 2.3472$$

$$y_5 = 1.2y_4 + (0.04)(4) = 1.2(2.3472) + (0.04)(4) = 2.97664$$

Exact solution

$$y' = (x + y)$$

$$y' - y = x$$

$$e^{-x} y' - y e^{-x} = x e^{-x}$$

$$d(y e^{-x}) = x e^{-x}$$

$$y e^{-x} = \int x e^{-x} dx = -x e^{-x} - e^{-x} + c$$

$$y e^{-x} = -x e^{-x} - e^{-x} + c$$

Substitute by the initial condition

$$y e^{-x} = -x e^{-x} - e^{-x} + c$$

$$1 = 0 - 1 + c \Rightarrow c = 2$$

$$\therefore y e^{-x} = x e^{-x} - e^{-x} + 2$$

$$y = -x - 1 + 2e^{x}$$

X=0.4

$$y = -0.4 - 1 + 2e^{-0.4} = 1.583649395$$

Error=**1.48 - 1.583649395**=0.103649395

Dr. Fathi Abdsallam